Oxford Cambridge and RSA

Monday 3 June 2019 – Morning

A Level Further Mathematics B (MEI)

Y420/01 Core Pure

Time allowed: 2 hours 40 minutes

You must have:
- Printed Answer Booklet
- Formulae Further Mathematics B (MEI)

You may use:
- a scientific or graphical calculator

INSTRUCTIONS

- Use black ink. HB pencil may be used for graphs and diagrams only.
- Answer **all** the questions.
- **Write your answer to each question in the space provided in the Printed Answer Booklet.** If additional space is required, you should use the lined page(s) at the end of the Printed Answer Booklet. The question number(s) must be clearly shown.
- You are permitted to use a scientific or graphical calculator in this paper.
- Final answers should be given to a degree of accuracy appropriate to the context.

INFORMATION

- The total number of marks for this paper is **144**.
- The marks for each question are shown in brackets **[ ]**.
- You are advised that an answer may receive **no marks** unless you show sufficient detail of the working to indicate that a correct method is used. You should communicate your method with correct reasoning.
- The Printed Answer Booklet consists of **24** pages. The Question Paper consists of **8** pages.

1 Find $\sum_{r=1}^{n}(2r^2-1)$, expressing your answer in fully factorised form. **[4]**

$$\sum_{r=1}^{n}(2r^2-1) = 2\sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} 1$$

$$= 2\times\frac{1}{6}n(n+1)(2n+1) - 1\times n$$

$$= \frac{1}{3}n(2n^2+3n+1) - n$$

$$= \frac{2}{3}n^3 + n^2 + \frac{1}{3}n - n$$

$$= \frac{1}{3}n(2n^2+3n-2)$$

$$= \frac{1}{3}n(2n-1)(n+2)$$

2 The plane $x+2y+cz=4$ is perpendicular to the plane $2x-cy+6z=9$, where c is a constant. Find the value of c. **[3]**

Perpendicular $\Rightarrow$ dot product of direction vectors $= 0$

$$\therefore\ (\underline{i}+2\underline{j}+c\underline{k}).(2\underline{i}-c\underline{j}+6\underline{k}) = 0$$

$$\Rightarrow 1(2)+2(-c)+c(6)$$

$$\Rightarrow 2-2c+6c=0$$

$$4c=-2$$

$$c=-\tfrac{1}{2}$$

3 Matrices **A** and **B** are defined by $\mathbf{A} = \begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix}k & 1\\ 2 & 0\end{pmatrix}$, where k is a constant.

(a) Verify the result $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$ in this case. **[5]**

(b) Investigate whether **A** and **B** are commutative under matrix multiplication. **[2]**

a. $\underline{A}\,\underline{B} = \begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}k & 1\\ 2 & 0\end{pmatrix} = \begin{pmatrix}3k+2 & 3\\ 2k+2 & 2\end{pmatrix}$

det = Red product
- Blue product

$\therefore \det(\underline{A}\,\underline{B}) = 6k+4-(6k+6) = -2$

$\therefore (\underline{A}\,\underline{B})^{-1} = -\frac{1}{2}\begin{pmatrix}2 & -3\\ -2k-2 & 3k+2\end{pmatrix}$

as the inverse of a matrix $= \frac{1}{\det}\begin{pmatrix}d & -b\\ -c & a\end{pmatrix}$

$A^{-1} = \frac{1}{3-2}\begin{pmatrix}1 & -1\\ -2 & 3\end{pmatrix} = \begin{pmatrix}1 & -1\\ -2 & 3\end{pmatrix}$

$B^{-1} = \frac{1}{-2}\begin{pmatrix}0 & -1\\ -2 & k\end{pmatrix}$

$\therefore \underline{B}^{-1}\underline{A}^{-1} = -\frac{1}{2}\begin{pmatrix}0 & -1\\ -2 & k\end{pmatrix}\begin{pmatrix}1 & -1\\ -2 & 3\end{pmatrix}$

$= -\frac{1}{2}\begin{pmatrix}2 & -3\\ -2-2k & 3k+2\end{pmatrix}$

$\therefore (\underline{A}\,\underline{B})^{-1} = \underline{B}^{-1}\underline{A}^{-1}$ as required

b. $\underline{A}\,\underline{B} = \begin{pmatrix} 3k+2 & 3 \\ 2k+2 & 2 \end{pmatrix}$

$\underline{B}\,\underline{A} = \begin{pmatrix} 3k+2 & k+1 \\ 6 & 2 \end{pmatrix}$

$\Rightarrow 2k+2 = 6 \quad \Rightarrow k = 2$

$k+1 = 3 \quad \Rightarrow k = 2$

$\underline{A}$ and $\underline{B}$ are commutative only when $k = 2$

4 **In this question you must show detailed reasoning.**

Fig. 4 shows the region bounded by the curve $y = \sec\frac{1}{2}x$, the x-axis, the y-axis and the line $x = \frac{1}{2}\pi$.

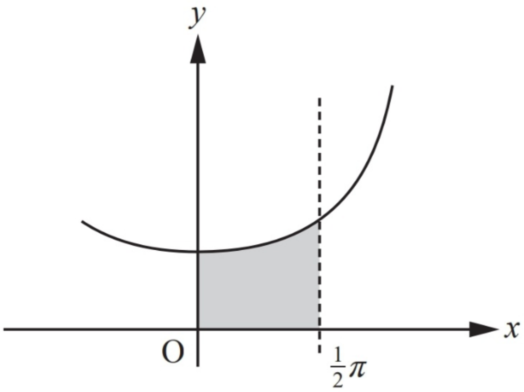


Fig. 4

This region is rotated through 2π radians about the x-axis.
Find, in exact form, the volume of the solid of revolution generated. **[3]**

$$V = \pi\int_a^b y^2\,dx$$

$$\therefore V = \pi\int_0^{\pi/2} \left(\sec^2 \tfrac{1}{2}x\right) dx$$

$$\left(\int \sec^2 x\,dx = \tan x \ (+c)\right)$$

$$= \pi\left[2\underbrace{\tan\tfrac{1}{2}x}_{\text{using chain rule}}\right]_0^{\pi/2}$$

$$= \pi\left(2\tan(\pi/4) - 0\right) = \underline{2\pi \text{ units}^2}$$

5 Using the Maclaurin series for $\cos 2x$, show that, for small values of x,

$\sin^2 x \approx ax^2 + bx^4 + cx^6$,

where the values of a, b and c are to be given in exact form. **[5]**

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \dots$$

$$= 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

(sub in $\cos^2 x = 1 - \sin^2 x$)

$$\cos 2x = 1 - 2\sin^2 x$$

$$\therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$= \frac{1}{2}\left(1 - 1 + 2x^2 - \frac{2}{3}x^4 + \frac{4}{45}x^6 + \dots\right)$$

$$= x^2 - \frac{1}{3}x^4 + \frac{2}{45}x^6 + \dots$$

so $a = 1$, $b = -\frac{1}{3}$, $c = \frac{2}{45}$

6 **In this question you must show detailed reasoning.**

Find $\int_{2}^{\infty} \frac{1}{4+x^2}\,\mathrm{d}x$. **[4]**

$I = \left[\frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) \right]_2^{\infty}$ one could also say $\arctan(x/2)$

as $x \to \infty$, $\tan^{-1}\left(\frac{x}{2}\right) \to \pi/2$

$\therefore\ I = \frac{1}{2} \times \frac{\pi}{2} - \frac{1}{2}\tan^{-1}(1)$

$= \frac{\pi}{4} - \frac{\pi}{8}$

$= \frac{\pi}{8}$

7 A curve has cartesian equation $(x^2+y^2)^2 = 2c^2xy$, where c is a positive constant.

(a) Show that the polar equation of the curve is $r^2 = c^2 \sin 2\theta$. **[2]**

(b) Sketch the curves $r = c\sqrt{\sin 2\theta}$ and $r = -c\sqrt{\sin 2\theta}$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$. **[3]**

(c) Find the area of the region enclosed by one of the loops in part **(b)**. **[3]**

a. $x^2 + y^2 = r^2$

$x = r\cos\theta$

$y = r\sin\theta$

becomes r^2

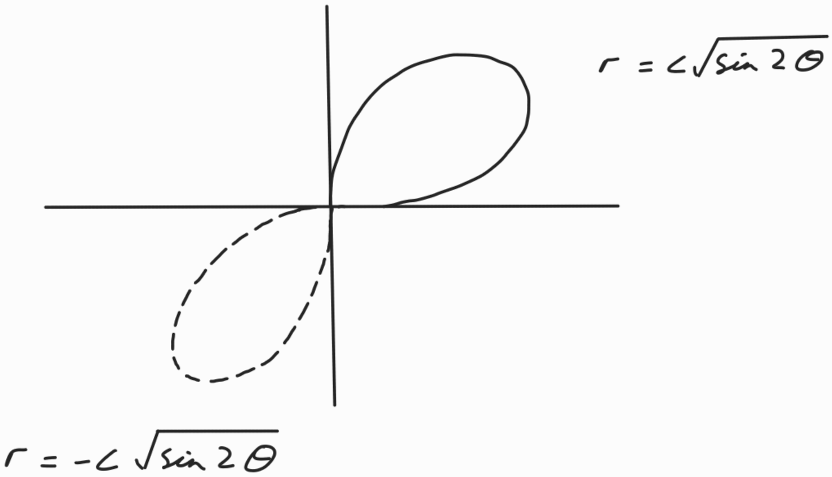
$\Rightarrow (r^2)^2 = 2c^2 r\cos\theta\, r\sin\theta$

$r^2 = 2c^2\cos\theta\sin\theta$ $(\div r^2)$

$r^2 = 2c^2 \sin 2\theta$ $(\sin 2\theta = \cos\theta \sin\theta)$

as required

b.



c. $A = \frac{1}{2}\int_a^b r^2 \, d\theta$

$$= \frac{1}{2}\int_0^{\pi/2} \left(c^2 \sin 2\theta \right) d\theta$$

$$= \left[-\frac{1}{4} c^2 \cos 2\theta \right]_0^{\pi/2}$$

$$= -\frac{1}{4} c^2 \cos \pi - \left(-\frac{1}{4} c^2 \cos 0 \right)$$

$$= \frac{1}{4} c^2 + \frac{1}{4} c^2$$

$$= \frac{1}{2} c^2$$

8 **In this question you must show detailed reasoning.**

The roots of the equation $x^3 - x^2 + kx - 2 = 0$ are $\alpha, \frac{1}{\alpha}$ and β.

(a) Evaluate, in exact form, the roots of the equation. **[6]**

(b) Find k. **[2]**

constant

a. product of roots $= \alpha \cdot \frac{1}{\alpha} \cdot \beta = -\frac{d}{a} = -\left(\frac{-2}{1}\right) = 2$

x^3 coefficient

$\Rightarrow \beta = 2$

x^2 coefficient

sum of roots $= \alpha + \frac{1}{\alpha} + \beta = -\frac{b}{a} = -\left(\frac{-1}{1}\right) = 1$

From above

$\Rightarrow \alpha + \frac{1}{\alpha} + \boxed{2} = 1$

$\alpha^2 + \alpha + 1 = 0$

$\alpha = \frac{-1 \pm \sqrt{3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

$\therefore$ roots are : $-\frac{1}{2} + \frac{\sqrt{3}}{2} i, -\frac{1}{2} - \frac{\sqrt{3}}{2} i, 2$

x coefficient

b. sum of products $= \alpha \cdot \frac{1}{\alpha} + \alpha\beta + \frac{1}{\alpha} \cdot \beta = c$

$\Rightarrow 1 + \alpha\beta + \frac{\beta}{\alpha} = k$

$1 + 2\left(\alpha + \frac{1}{\alpha}\right) = k$

$1 + 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i - \frac{1}{2} + \frac{\sqrt{3}}{2} i\right) = k$

$k = 1 - 2 = \underline{-1}$

9 Prove by induction that $5^n + 2 \times 11^n$ is divisible by 3 for all positive integers n. **[7]**

$\underline{n=1}$:

$u_1 = 5 + 2 \times 11 = 27$

$\therefore$ True for $n = 1$

$\underline{\text{Assume true for } n = k}$:

$\Rightarrow u_k \; 5^k + 2 \times 11^k$ is divisible by 3

$\underline{n = k+1}$:

$5^{k+1} + 2 \times 11^{k+1}$

$= 5(5^k) + 22 \times 11^k$

$= 5(u_k - 2 \times 11^k) + 22 \times 11^k$

$= 5u_k - 10 \times 11^k + 22 \times 11^k$

$= 5u_k + 12 \times 11^k$

12 is div by 3

div by 3 as u_k is div by 3

$\therefore$ Statement is true for $n = 1$, and true for $n = k+1$ when assumed true for $n = k$. So true for all positive integers n.

10 In this question you must show detailed reasoning.

(a) You are given that $-1+\mathrm{i}$ is a root of the equation $z^3 = a+b\mathrm{i}$, where a and b are real numbers. Find a and b. **[3]**

(b) Find all the roots of the equation in part **(a)**, giving your answers in the form $re^{\mathrm{i}\theta}$, where r and θ are exact. **[4]**

(c) Chris says "the complex roots of a polynomial equation come in complex conjugate pairs". Explain why this does **not** apply to the polynomial equation in part **(a)**. **[1]**

a. $\Rightarrow z_1 = -1+i$

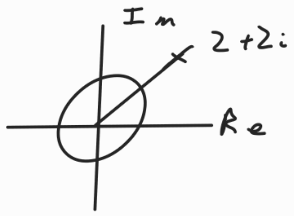
$$\therefore \; z^3 = (-1+i)^3 = (-1)^3 + 3(-1)^2 i + 3(-1)i^2 + i^3$$

$$= -1 + 3i + 3 - i$$

$$= 2 + 2i \qquad \therefore a = 2 \; , \; b = 2$$

b. $z^3 = 2+2i$

$$r = |z^3| = \sqrt{2^2+2^2} = 2\sqrt{2}$$



$$\arg(z^3) = \frac{9\pi}{4}$$

$n = 3 \; , \; k = -2, -1, 0$

$$z_1 = (2\sqrt{2})^{\frac{1}{3}} e^{\left(\frac{\frac{9\pi}{4} - 4\pi}{3}\right)i} = \sqrt{2}\, e^{\frac{-7i\pi}{12}}$$

$$z_2 = (2\sqrt{2})^{\frac{1}{3}} e^{\left(\frac{\frac{9\pi}{4} - 2\pi}{3}\right)i} = \sqrt{2}\, e^{\frac{i\pi}{12}}$$

$$z_3 = (2\sqrt{2})^{\frac{1}{3}} e^{\left(\frac{\frac{9\pi}{4} + 0}{3}\right)i} = \sqrt{2}\, e^{\frac{3i\pi}{4}}$$

c. This only applies to polynomials with real coefficients.

11 **(a)** Specify fully the transformations represented by the following matrices.

- $\mathbf{M}_1 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$

- $\mathbf{M}_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **[4]**

(b) Find the equation of the mirror line of the reflection R represented by the matrix $\mathbf{M}_3 = \mathbf{M}_1\mathbf{M}_2$. **[5]**

(c) It is claimed that the reflection represented by the matrix $\mathbf{M}_4 = \mathbf{M}_2\mathbf{M}_1$ has the same mirror line as R. Explain whether or not this claim is correct. **[3]**

a. $\cos\theta = 3/5$ and $\sin\theta = 4/5$

$\therefore \theta = \cos^{-1}(3/5) = 53.1°$

M_1 is a rotation anti-clockwise about O through $\underbrace{53.1°}_{\cos^{-1}(3/5)}$

M_2 is a reflection in the x-axis

b. $M_3 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

Apply M_3 to $\begin{pmatrix} x \\ y \end{pmatrix}$: $\begin{pmatrix} \frac{3}{5} & \frac{4}{5} \\ \frac{4}{5} & -\frac{3}{5} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$

$\Rightarrow \frac{3}{5}x + \frac{4}{5}y = x$

$\frac{4}{5}y = \frac{2}{5}x$

$y = \frac{1}{2}x \quad \therefore \underline{y = \frac{1}{2}x}$ is the mirror line

c. $M_4 = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & -\frac{3}{5} \end{pmatrix}$

$M_4 \neq M_3$, so they do not represent the same reflection, and cannot have the same mirror line.

12 Three intersecting lines L_1, L_2 and L_3 have equations

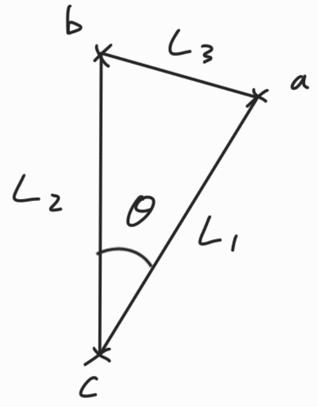
$$L_1: \frac{x}{2} = \frac{y}{3} = \frac{z}{1}, \quad L_2: \frac{x}{1} = \frac{y}{2} = \frac{z}{-4} \quad \text{and} \quad L_3: \frac{x-1}{1} = \frac{y-2}{1} = \frac{z+4}{5}.$$

Find the area of the triangle enclosed by these lines. **[9]**

$L_1: r = \lambda\left(2\underline{i} + 3\underline{j} + \underline{k}\right)$

$L_2: r = \alpha\left(\underline{i} + 2\underline{j} - 4\underline{k}\right)$

$L_3: r = \underline{i} + 2\underline{j} - 4k + v\left(\underline{i} + \underline{j} + 5k\right)$



a: L_1 and L_3:

Equate components of:

$i: 2\lambda = 1 + v$ ①

$j: 3\lambda = 2 + v$ ②

$k: \lambda = -4 + 5v$ ③

② - ①: $3\lambda - 2\lambda = 2 + v - (1 + v)$

$\lambda = 1$

Sub $\lambda = 1$ into ③ : $1 = -4 + 5v$

$v = 1$

Sub $\lambda = 1$ into L_1, or $v = 1$ into L_3 for point of intersection :

$(2, 3, 1)$

b: L_2 and L_3 :

Equate coefficient :

i. $\alpha = 1 + v$ ①

j. $2\alpha = 2 + v$ ②

k. $-4\alpha = -4 + 5v$ ③

② - ① : $2\alpha - \alpha = 2 + v - (1 + v)$

$\alpha = 1$

Sub $\alpha = 1$ into ③ : $-4 = -4 + 5v$

$v = 0$

Sub $\alpha = 1$ into L_2, or $v = 0$ into L_3 for point of intersection :

$(1, 2, -4)$

c: L_1 and L_2 :

Equate coefficients :

i. $2\lambda = \alpha$ ①

j. $3\lambda = 2\alpha$ ②

k. $\lambda = -4\alpha$ ③

② $- 2 \times$ ①: $3\lambda - 4\lambda = 2\alpha - 2\alpha$

$$\lambda = 0$$

$$\Rightarrow \alpha = 0$$

Sub $\lambda = 0$ into L_1, or $\alpha = 0$ into L_2 for point of intersection:

$(0, 0, 0)$ or origin.

θ:

$$\cos\theta = \frac{\underline{a}.\underline{b}}{|\underline{a}||\underline{b}|}$$ (using L_1 and L_2 as per diagram)

$$\cos\theta = \frac{(2\underline{i} + 3\underline{j} + \underline{k}).(\underline{i} + 2\underline{j} - 4\underline{k})}{\sqrt{2^2 + 3^2 + 1^2} \times \sqrt{1^2 + 2^2 + (-4)^2}}$$

$$= \frac{2 + 6 - 4}{\sqrt{14}\sqrt{21}}$$

$$\cos\theta = \frac{4}{\sqrt{294}}$$

$$\Rightarrow \theta = 76.5^0$$

$\therefore$ area $= \frac{1}{2} ab \sin C$

$$= \frac{1}{2} \times \sqrt{14} \times \sqrt{21} \times \sin 76.5$$

$$= 8.34 \text{ units}^2$$

13 **(a)** Using the logarithmic form of arcosh x, prove that the derivative of arcosh x is $\frac{1}{\sqrt{x^2-1}}$. **[5]**

(b) Hence find $\int_1^2 \operatorname{arcosh} x \, dx$, giving your answer in exact logarithmic form. **[5]**

(c) Ali tries to evaluate $\int_0^1 \operatorname{arcosh} x \, dx$ using his calculator, and gets an 'error'. Explain why. **[1]**

a. $y = \operatorname{arcosh} x = \ln\left(x + \sqrt{x^2-1}\right)$

$e^y = x + \sqrt{x^2-1} = x + (x^2-1)^{1/2}$

chain rule

$\therefore e^y \frac{dy}{dx} = 1 + x(x^2-1)^{-1/2}$

$\therefore$ as $e^y = x + (x^2-1)^{1/2}$,

$$\frac{dy}{dx} = \frac{1 + x(x^2-1)^{-1/2}}{x + (x^2-1)^{1/2}} \qquad \left(\times \frac{x - (x^2-1)^{1/2}}{x - (x^2-1)^{1/2}}\right)$$

$$\frac{dy}{dx} = \frac{\cancel{x} + x^2(x^2-1)^{-1/2} - (x^2-1)^{1/2} \cancel{- x}}{\cancel{x^2} + \cancel{x(x^2-1)^{1/2}} - \cancel{x(x^2-1)^{1/2}} - (\cancel{x^2}-1)}$$

$$\frac{dy}{dx} = \frac{x^2}{(x^2-1)^{1/2}} - (x^2-1)^{1/2} \qquad \left(\times \text{ 2nd term by } \frac{(x^2-1)^{1/2}}{(x^2-1)^{1/2}}\right)$$

$$\frac{dy}{dx} = \frac{x^2}{(x^2-1)^{1/2}} - \frac{(x^2-1)}{(x^2-1)^{1/2}}$$

$$\frac{dy}{dx} = \frac{x^2 - x^2 + 1}{(x^2-1)^{1/2}} = \frac{1}{(x^2-1)^{1/2}}$$ as required

b. Let $u = \operatorname{arcosh} x$ $\qquad \frac{dv}{dt} = 1$

$\frac{du}{dx} = \frac{1}{\sqrt{x^2-1}}$ $\qquad v = x$

By parts: $uv - \int v \frac{du}{dx}\, dx$

$$\int_1^2 (\operatorname{arcosh} x)\, dx = [x \operatorname{arcosh} x]_1^2 - \int_1^2 \left(\frac{x}{\sqrt{x^2-1}}\right) dx$$

$$= \left[x \operatorname{arcosh} x - \sqrt{x^2-1}\right]_1^2$$

$$= 2 \operatorname{arcosh} 2 - \sqrt{3} - (\operatorname{arcosh} 1 - \sqrt{0})$$

$$= 2 \ln(2+\sqrt{3}) - \sqrt{3}$$

c. arcosh x does not exist for $x < 1$

14 Three planes have equations

$$\begin{aligned} -x+ay \quad &= 2 \\ 2x+3y+z &= -3 \\ x+by+z &= c \end{aligned}$$

where a, b and c are constants.

(a) In the case where the planes **do not** intersect at a unique point,

(i) find b in terms of a, **[4]**

(ii) find the value of c for which the planes form a sheaf. **[3]**

(b) In the case where $b = a$ and $c = 1$, find the coordinates of the point of intersection of the planes in terms of a. **[6]**

a. i. Let $\underline{M} = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & b & 1 \end{pmatrix}$

Planes do not have a unique point of intersection
$\Rightarrow \det \underline{M} = 0$

$$\det \underline{M} = -1(3-b) - a(2-1) + 0 = 0$$
$$= -3 + b - a = 0$$

$$\Rightarrow b = a + 3$$

ii. Sheaf $\Rightarrow$ equations are consistent

$-x + ay = 2$
$x = ay - 2$ ①

Sub ① into : $2x + 3y + z = -3$:
$(2a+3)y + z = 1$

Sub ① into : $x + by + z = c$

$$(a+b)y + z = c+2$$

$$(2a+3)y + z = c+2 \qquad (b = a+3) \text{ from i.}$$

$$\Rightarrow c + 2 = 1$$

$$c = -1$$

6. Points of intersection given by $\underline{M}^{-1}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$\underline{M} = \begin{pmatrix} -1 & a & 0 \\ 2 & 3 & 1 \\ 1 & a & 1 \end{pmatrix}$$

$$\det \underline{M} = -1(3-a) - a(2-1) + 0$$
$$= -3$$

matrix of minors : $\begin{pmatrix} 3-a & 1 & 2a-3 \\ a & -1 & -2a \\ a & -1 & -3-2a \end{pmatrix}$

matrix of cofactors : $\begin{pmatrix} 3-a & -1 & 2a-3 \\ -a & -1 & 2a \\ a & 1 & -3-2a \end{pmatrix}$

adjugate : $\begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$

$$\therefore \underline{M}^{-1} = -\frac{1}{3}\begin{pmatrix} 3-a & -a & a \\ -1 & -1 & 1 \\ 2a-3 & 2a & -3-2a \end{pmatrix}$$

Points of intersection $= \underline{M}^{-1}\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$

$$= -\frac{1}{3}\begin{pmatrix} 6-2a+3a+a \\ -2+3+1 \\ 4a-6-6a-3-2a \end{pmatrix}$$

$$= -\frac{1}{3}\begin{pmatrix} 6+2a \\ 2 \\ -4a-9 \end{pmatrix}$$

$\therefore$ coordinates are: $\left(-\frac{6+2a}{3}, -\frac{2}{3}, \frac{4a+9}{3}\right)$

15 In this question you must show detailed reasoning.

Show that $\int_{\frac{3}{4}}^{\frac{3}{2}} \frac{1}{\sqrt{4x^2-4x+2}}\,\mathrm{d}x = \frac{1}{2}\ln\left(\frac{3+\sqrt{5}}{2}\right)$. **[8]**

$$4x^2-4x+2 = 4x^2-4x+1+1$$
$$= (2x-1)^2+1$$

$$\Rightarrow \int_{\frac{3}{4}}^{\frac{3}{2}}\left(\frac{1}{\sqrt{(2x-1)^2+1}}\right)dx$$

as $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)$,

due to $2x$

$$\Rightarrow \int_{\frac{3}{4}}^{\frac{3}{2}}\left(\frac{1}{\sqrt{(2x-1)^2+1}}\right)dx = \frac{1}{2}\left[\sinh^{-1}(2x-1)\right]_{3/4}^{3/2}$$
$$= \frac{1}{2}\left(\sinh^{-1}(2) - \sinh^{-1}\left(\tfrac{1}{2}\right)\right)$$
$$= \frac{1}{2}\left(\ln(2+\sqrt{5}) - \ln(\tfrac{1}{2}+\sqrt{5/4}\right)$$
$$= \frac{1}{2}\ln\left(\frac{2+\sqrt{5}}{\frac{1}{2}+\frac{\sqrt{5}}{2}}\right)$$
$$= \frac{1}{2}\ln\left(\frac{4+2\sqrt{5}}{1+\sqrt{5}}\right)$$

(rationalise)

$$= \frac{1}{2}\ln\left(\frac{(4+2\sqrt{5})(\sqrt{5}-1)}{(1+\sqrt{5})(\sqrt{5}-1)}\right)$$

$$= \frac{1}{2} \ln \left(\frac{2\sqrt{5} + 10 - 4}{5 - 1} \right)$$

$$= \frac{1}{2} \ln \left(\frac{\sqrt{5} + 3}{2} \right)$$ as required

16 **(a)** Show that $(2-\mathrm{e}^{\mathrm{i}\theta})(2-\mathrm{e}^{-\mathrm{i}\theta}) = 5-4\cos\theta$. **[3]**

Series C and S are defined by

$$C = \frac{1}{2}\cos\theta + \frac{1}{4}\cos 2\theta + \frac{1}{8}\cos 3\theta + \ldots + \frac{1}{2^n}\cos n\theta,$$

$$S = \frac{1}{2}\sin\theta + \frac{1}{4}\sin 2\theta + \frac{1}{8}\sin 3\theta + \ldots + \frac{1}{2^n}\sin n\theta.$$

(b) Show that $C = \dfrac{2^n(2\cos\theta - 1) - 2\cos(n+1)\theta + \cos n\theta}{2^n(5-4\cos\theta)}$. **[9]**

a. Expand:

$$(2 - e^{i\theta})(2 - e^{-i\theta}) = 4 - 2e^{i\theta} - 2e^{-i\theta} + 1$$

Sub in $e^{i\theta} + e^{-i\theta} = 2\cos\theta$:

$$4 - 2(e^{i\theta} + e^{-i\theta}) + 1 = 5 - 4\cos\theta$$

b. $C + iS = \frac{1}{2}e^{i\theta} + \frac{1}{4}e^{2i\theta} + \frac{1}{8}e^{3i\theta} + \ldots + \frac{1}{2^n}e^{ni\theta}$

This is a geometric series, $a = \frac{1}{2}e^{i\theta}$

$r = \frac{1}{2}e^{i\theta}$

$\therefore$ $C + iS$ = sum to infinity

$$= \frac{\frac{1}{2}e^{i\theta}\left(1-\left(\frac{1}{2}e^{i\theta}\right)^n\right)}{1-\frac{1}{2}e^{i\theta}}$$

$$= \frac{e^{i\theta}\left(1-\left(\frac{1}{2}e^{i\theta}\right)^n\right)}{2-e^{i\theta}} \qquad \left(\times \frac{2+e^{i\theta}}{2+e^{i\theta}}\right)$$

$$= \frac{e^{i\theta}\left(2-\frac{1}{2^{n-1}}-e^{-i\theta}+\frac{1}{2^n}e^{(n-1)i\theta}\right)}{5-4\cos\theta} \qquad \left(\times 2^n/2^n\right)$$

$$C+iS = \frac{2^{n+1}e^{i\theta}-2e^{(n+1)i\theta}-2^n+e^{ni\theta}}{2^n(5-4\cos\theta)}$$

$\Rightarrow$ C = real parts :

$$= \frac{2^n(2\cos\theta-1)-2\cos(n+1)\theta+\cos n\theta}{2^n(5-4\cos\theta)}$$

as required

17 A cyclist accelerates from rest for 5 seconds then brakes for 5 seconds, coming to rest at the end of the 10 seconds. The total mass of the cycle and rider is m kg, and at time t seconds, for $0 \leqslant t \leqslant 10$, the cyclist's velocity is $v\,\mathrm{m\,s^{-1}}$.

A resistance to motion, modelled by a force of magnitude $0.1mv$ N, acts on the cyclist during the whole 10 seconds.

(a) Explain why modelling the resistance to motion in this way is likely to be more realistic than assuming this force is constant. **[1]**

During the braking phase of the motion, for $5 \leqslant t \leqslant 10$, the brakes apply an additional constant resistance force of magnitude $2m$ N and the cyclist does not provide any driving force.

(b) Show that, for $5 \leqslant t \leqslant 10$, $\frac{\mathrm{d}v}{\mathrm{d}t} + 0.1v = -2$. **[1]**

(c) **(i)** Solve the differential equation in part **(b)**. **[5]**

(ii) Hence find the velocity of the cyclist when $t = 5$. **[1]**

During the acceleration phase $(0 \leqslant t \leqslant 5)$, the cyclist applies a driving force of magnitude directly proportional to t.

(d) Show that, for $0 \leqslant t \leqslant 5$, $\frac{\mathrm{d}v}{\mathrm{d}t} + 0.1v = \lambda t$, where λ is a positive constant. **[1]**

(e) **(i)** Show by integration that, for $0 \leqslant t \leqslant 5$, $v = 10\lambda(t - 10 + 10\mathrm{e}^{-0.1t})$. **[5]**

(ii) Hence find λ. **[2]**

(f) Find the total distance, to the nearest metre, travelled by the cyclist during the motion. **[6]**

a. The resistance force is likely to increase with velocity.

b. By $f = ma$:

$$-2m - 0.1mv = m\frac{dv}{dx}$$

$$\Rightarrow \frac{dv}{dx} + 0.1v = -2$$

as required.

c. i. $IF = e^{0.1t}$

$$\Rightarrow \frac{d}{dt}(ve^{0.1t}) = -2e^{0.1t}$$

$$\Rightarrow ve^{0.1t} = -2\int e^{0.1t}\,dt$$

$$ve^{0.1t} = -20e^{0.1t} + c$$

when $t = 10$, $v = 0$ $\Rightarrow c = 20e$

$$\Rightarrow ve^{0.1t} = -20e^{0.1t} + 20e$$

$$v = -20 + 20e^{(1-0.1t)}$$

$$v = 20(e^{1-0.1t} - 1)$$

ii. at $t = 5$, $v = 20(e^{0.5} - 1)$

$= 12.97\,ms^{-1}$ to 2dp.

d. By $f = ma$:

$$ct - 0.1mv = m\frac{dv}{dt}$$

$(\lambda = c/m)$

$\therefore \frac{dv}{dt} + 0.1v = \lambda t$ as required

e. i. If $= e^{0.1t}$

$$\Rightarrow \frac{d}{dt} ve^{0.1t} = \lambda t e^{0.1t}$$

$$\therefore \; ve^{0.1t} = \int \lambda t e^{0.1t} dt$$

By parts : $u = t$ $\quad \frac{dv}{dt} = \lambda e^{0.1t}$

$\frac{du}{dt} = 1$

$v = 10\lambda e^{0.1t}$

$$\Rightarrow ve^{0.1t} = 10\lambda t e^{0.1t} - \int 10\lambda e^{0.1t} dt$$

$$ve^{0.1t} = 10\lambda t e^{0.1t} - 100\lambda e^{0.1t} + c$$

when $t = 0$, $v = 0 \Rightarrow c = 100\lambda$

$$\Rightarrow v\cancel{e^{0.1t}} = 10\lambda(t \cancel{e^{0.1t}} - 10\cancel{e^{0.1t}} + 10)$$

$$v = 10\lambda(t - 10 + 10e^{-0.1t})$$

ii. As both ranges, $5 \leq t \leq 10$ and $0 \leq t \leq 5$ include 5 seconds, the speeds at $t = 5$ must be equal.

$$\therefore \; v = 20(e^{0.5} - 1) = 10\lambda(10e^{-0.5} - 5)$$

$$\lambda = \frac{20(e^{0.5} - 1)}{10(10e^{-0.5} - 5)} = 1.218$$

f. $0 \leq t \leq 5$:

$$s_1 = \int_0^5 (10\lambda(t - 10 + 10e^{-0.1t}) \, dt$$

$$= 10\lambda\left[\frac{1}{2}t^2 - 10t - 100e^{-0.1t} \right]_0^5$$

$$= 12.18(12.5 - 50 - 100e^{-0.5})$$

$$= 22.49 m$$

$5 \leq t \leq 10$:

$$s_2 = \int_5^{10} 20(e^{1-0.1t} - 1) \, dt$$

$$= 20\left[-10e^{1-0.1t} - t \right]_5^{10}$$

$$= 20(-10e^0 - 10 + 10e^{0.5} + 5)$$

$$= 20(-15 + 10e^{0.5})$$

$$= 29.74 m$$

$\therefore$ Total distance : 52 m